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Prepared by

CATHOLIC UNIVERSITY

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A SURVEY OF STRUCTURAL DYNAMICS OF SOLID PROPELLANT ROCKET MOTORS*

by J. H. Baltrukonis

ABSTRACT

From the point of view of dynamics, a solid propellant rocket motor is an unique structure in that it is composed of a substantial mass of propellant material case-bonded to a relatively massless, thin-walled cylinder. The mechanical properties of the propellant are such that it contributes little to the stiffness of the composite structure but it does contribute to the dynamic characteristics of the structure because of its mass. Furthermore, due to the viscoelastic character of the propellant, it can be expected to provide considerable damping to the system. At this point in the development of the art of design of solid propellant rocket motors there are no clear-cut or well-founded methods to quantitatively evaluate the contributions of the propellant to the dynamic responses of the composite structure. It is clear that unless such methods are devised, it will be difficult to arrive at accurate missile designs.

In the paper a survey of the dynamic problems of solid propellant rocket motors is presented starting from the simplest model thereof and proceeding, step-by-step, to the consideration of more sophisticated and realistic models. The consideration is restricted to infinitesimal deformations of propellant grains with linear mechanical properties. Substantial progress has been achieved towards the solution of many important dynamical problems. We shall attempt to summarize the pertinent developments and indicate along which lines we feel future study should proceed. A few illustrative solutions are included in areas wherein progress has been substantial.

*An abbreviated form of this paper was presented at the International Conference on the Mechanics and Chemistry of Solid Propellants held at Purdue University, Lafayette, Indiana on April 19-21, 1965.

INTRODUCTION

Structural dynamics concerns the analysis, by theoretical and/or experimental means, of the interactions of time-dependent loads and/or deformations externally applied to a structure or structural element and the internal stress and displacement response wherein inertial effects must be included in the analysis. It is the objective of this paper to present a survey of the field of structural dynamics of solid propellant rocket motors, to discuss those aspects of the subject which are of particular interest to the author and to recommend those areas in which further study should prove fruitful and rewarding. It is not our objective to present a bibliographical survey of the field and, consequently, many specific, individual contributions will probably be overlooked. There is no claim of uniqueness for this survey of the field nor do we maintain that it is complete since it is inevitable that a study of this sort will be biased to a large extent by the limitations, interests and viewpoint of the author.

The first logical step in a task of this sort is to delimit the field under consideration. In general, we shall not be interested in the dynamics of complete rockets or missiles which may consist of several stages. Instead, we shall limit our concern to individual solid propellant rocket motors; i.e., casing plus propellant. We shall not be interested in any attachments to the casing such as rocket nozzles, guidance and control system assemblies, etc. It is not intended to imply that the dynamics of complete rockets is not important, but rather the intention is to limit the scope of this presentation.

A solid propellant rocket motor is an unusually complicated structure, at least from the standpoint of analysis thereof. It consists of a thin, circular cylindrical casing with domed end closure. The energetic propellant grain is bonded to the casing along its outer cylindrical surface. Frequently, a rubber liner is interposed between the grain and casing for purposes of insulation. The flow of the gases developed by surface combustion of the solid propellant occurs through passages of relatively complex geometry within the grain to the one or more nozzles in the aft dome of the motor. The motor casing material is usually an elastic material such as steel though there has been a recent trend to casings wound of fiberglass filaments and impregnated with some sort of hard resin.

The propellant material is a composite consisting of an elastomeric binder, a crystalline oxidizer and dispersed solid materials such as aluminum particles. This material is very compliant relative to the casing, is time- or frequency-dependent and is highly temperature-sensitive. Finally, the grain is very massive compared to the casing usually constituting from 80 to 95 per cent of the total mass of the motor.

Clearly, analysis of this structure for its responses to various dynamic stimuli is an imposing task indeed. The usual first step in the analysis of

a complicated structure is the formulation of a tractable mathematical model. The structure involves several fundamental difficulties but probably the principal ones are (1) the complex geometry of the internal passages in the propellant grain; and (2) the fact that the motor is of finite length thereby introducing possible interaction of end effects. These two difficulties are of the same nature as the complications that have plagued elasticians from the very beginnings of the field of elasticity. Consequently, at the outset, it seems wise to idealize the structure to the extent that these complications are no longer present. Thus, we consider a mathematical model that is infinitely-long with a circular perforation. We do not include a liner so that the motor consists only of grain and casing. We further restrict the present consideration to linear analysis; i.e., we consider only infinitesimal deformations of linear materials. These restrictions may not be as severe as they seem when it is realized that most of the dynamic environments of interest will result in very small displacements within the linear range of the materials under consideration. Materials properties investigations have revealed that, for small strains, propellants typically behave as linearly-viscoelastic solids.

ELASTIC GRAINS

The mathematical model which we have thus far devised is still too complicated for initial studies. Thus, we introduce the further assumptions that the grain is perfectly elastic with no internal perforation and, since the casing is very stiff relative to the core, that the casing be perfectly rigid. Therefore, our initial, very crude model of a solid propellant rocket motor consists of an infinitely-long composite cylinder with a rigid outer layer and a very compliant, solid, elastic core. Investigation of such a model, however crude, will begin to yield valuable information concerning natural frequencies and displacement and stress fields corresponding to normal modes. Such information is of immediate utility, for example, in the design of the guidance systems whose reliability depends, to a large extent, on estimates of natural frequencies and mode shapes of the rocket motor.

Other possible applications are dynamic loads analyses, staging studies, transportation and handling studies, etc. Over and above these practical applications, initial analyses of crude mathematical models provides the analyst with the experience required for subsequent refinement of the model. Additionally, a catalog of solutions is rapidly developed which may prove to be very useful as checks on the solutions of more refined models.

The natural coordinate system for the formulation of the problem posed is the polar cylindrical system. There has long been activity within the general area of dynamics of elastic bodies in polar cylindrical coordinates. The earliest formulation of a problem by means of the dynamical equations of elasticity in cylindrical coordinates is due to Pochhammer (1) and Chree (2) who independently investigated longitudinal and transverse wave propagation in infinitely-long circular rods with traction-free surfaces. An excellent

survey of the history of this problem was presented by Abramson, Plass and Ripperger (3) but, nevertheless, it would be useful and interesting to mention a few of the more important investigations. Following Pochhammer and Chree the next study that should be mentioned is due to Ghosh (4) who derived dispersion equations for longitudinal wave propagation in thick- and thin-walled, infinitely-long, hollow, circular cylinders with both surfaces free of traction and with one surface traction-free and the other rigidly clamped.

Bancroft (5) was among the first to publish numerical solutions of the dispersion equations for longitudinal wave propagation. In addition, he recorded the dispersion equation for transverse wave propagation in infinitely-long, circular rods with traction-free surfaces but no numerical work was indicated. Hudson (6) carried out some numerical work with the dispersion equation for transverse wave propagation but arrived at the erroneous conclusion that there was only one mode of propagation of transverse waves. This error was subsequently propagated by Davies (7) and Kolsky (8), among others, and was finally pointed out by Abramson (9) who calculated several dispersion curves in the first mode of propagation of transverse waves.

McFadden (10) was concerned with radial vibrations in hollow, thick-walled cylinders while Gazis (11) presented a study of plane strain vibrations in hollow cylinders with traction-free surfaces. The most complete study, to date, by means of the dynamical equations of elasticity, was presented by Gazis (12) on the longitudinal and transverse wave propagation and free vibrations in infinitely-long, thick-walled cylinders with traction-free surfaces. Other pertinent studies were reported by Herrmann and Mirsky (13), Mirsky and Herrmann (14, 15), Greenspon (16, 17, 18) Bird, Hart and McClure (19) and Bird (20).

The field and constitutive equations for dynamic deformations of compressible elastic continua in polar cylindrical coordinates may be reduced to the following three equations of motion:

$$\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{1-2\nu}\frac{\partial\Delta}{\partial r} = \frac{\rho}{G}\frac{{}^2 u_r}{\partial t^2} \tag{1a}$$

$$\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} + \frac{1}{1-2\nu}\frac{1}{r}\frac{\partial\Delta}{\partial\theta} = \frac{\rho}{G}\frac{\partial^2 u_\theta}{\partial t^2} \tag{1b}$$

$$\nabla^2 u_z + \frac{1}{1-2\nu}\frac{\partial\Delta}{\partial z} = \frac{\rho}{G}\frac{\partial u_z}{\partial t^2} \tag{1c}$$

Solutions of these equations of motion are readily obtained by means of three displacement potentials as follows:

$$u_r = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r} \frac{\partial \chi}{\partial \theta} \tag{2a}$$

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial z} - \frac{\partial \chi}{\partial r} \tag{2b}$$

$$u_z = \frac{\partial \phi}{\partial z} - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \psi . \tag{2c}$$

It may be verified by direct substitution that Eqs. (1) are identically satisfied by these components of displacement provided we take the displacement potentials as solutions of the following differential equations:

$$\nabla^2 \phi = c_c^{-2} \frac{\partial^2 \phi}{\partial t^2} \tag{3a}$$

$$\nabla^2 (\psi, \chi) = c_s^{-2} \frac{\partial^2}{\partial t^2} (\psi, \chi), \tag{3b}$$

wherein c_c and c_s denote the dilatational and shear wave velocities, respectively, and are given by

$$c_c^2 = \kappa^2 (G/\rho) \tag{4a}$$

$$c_s^2 = G/\rho \tag{4b}$$

$$\kappa = C_s/C_c = [(1-2\nu)\ /\ 2(1-\nu)]^{1/2} \tag{4c}$$

We recognize the differential equations (3) as wave equations, solutions of which are well known. On selection of appropriate solutions of these wave equations, the displacements follow from Eqs. (2) while the components of stress are obtained from the following stress-displacement relations:

$$\sigma_r = 2G\left(\frac{\partial u_r}{\partial r} + \frac{\lambda}{2G}\Lambda\right) \tag{5a}$$

$$\sigma_\theta = 2G\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\lambda}{2G}\Lambda\right) \tag{5b}$$

$$\sigma_z = 2G\left(\frac{\partial u_z}{\partial z} + \frac{\lambda}{2G}\Lambda\right) \tag{5c}$$

$$\tau_{r\theta} = G\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \tag{5d}$$

$$\tau_{rz} = G\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{5e}$$

$$\tau_{\theta z} = G\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \tag{5f}$$

Let us now illustrate the application of this general theory in the calculation of the natural frequencies and normal modes of the initial crude model for the solid propellant rocket motor. Since the grain is solid, the boundary conditions are given by

$$u_r\Big|_{r=a} = u_\theta\Big|_{r=a} = u_z\Big|_{r=a} = 0 \tag{6}$$

The simplest modes of vibration are the so-called axial-shear modes which are defined as those modes of free vibration in which the only non-zero component of displacement is that parallel to the axis of the cylinder. Furthermore, this axial displacement component depends only on the coordinates in the plane of a cross-section. These modes are of fundamental importance if for no other reason than that they are among the very few modes for which exact, closed-form solutions of the three-dimensional equations of elasticity are possible. There are, however, other reasons for concern with axial-shear modes. They represent limiting flexural or longitudinal modes; i.e., they are flexural or longitudinal wave modes with infinite wave-length. In accord with their definition we seek solutions of the equations of elasticity in the form

$$u_r = u_\theta = 0 \tag{7a}$$

$$u_z = W(r)\, e^{i\omega t} \cos n\theta \tag{7b}$$

Substitution into Eqs. (1) results in the following single equaiton:

$$W'' + \frac{1}{r} W' + \left(\frac{\rho\omega^2}{G} - \frac{n^2}{r^2} \right) W = 0 \tag{8}$$

wherein primes denote differentiation with respect to the argument. This is the well-known n^{th}-order Bessel equation which has the following general solution:

$$W(r) = C_{n1}\, J_n\, (\Omega r/a) + C_{n2}\, Y_n\, (\Omega\, r/a) \tag{9a}$$

wherein Ω is a dimensionless frequency coefficient defined by

$$\Omega^2 = \frac{\rho\omega^2 a^2}{G} \tag{9b}$$

C_{n1} and C_{n2} are arbitrary constants which must be evaluated such that the boundary conditions will be satisfied, and J_n and Y_n are the n^{th}-order Bessel functions of the 1st and 2nd kinds, respectively. In view of Eqs. (7) the boundary conditions given by Eqs. (6) reduce to the following single condition:

$$W(a) = 0$$

This condition will be identically satisfied provided that we take C_{2n} to vanish and that

$$J_n(\Omega) = 0 \tag{10}$$

This result constitutes the frequency equation in the present problem. It defines a doubly-infinite family of natural frequencies and normal modes; i.e., corresponding to each value of n we find an infinite number of roots of Eq. (10). It is clear from Eq. (9a) that the n=0 modes are axisymmetric whereas the modes for which n=1 are antisymmetric. In view of Eqs. (9a) and (7), we observe that all the stresses vanish with the exception of the shear stresses τ_{rz} and $\tau_{\theta r}$ and the latter vanishes for the axisymmetric modes. Clearly, this general type of vibration is pure shear in nature accounting, partially, for the name axialshear. These modes, among others, were thoroughly investigated in Ref. (21).

Another important class of vibrations occurs when the axial component of displacement vanishes identically yielding a transverse mode of vibration. Such a mode of vibration occurs when we take

$$\phi = C_{n1} J_n (\kappa\Omega\frac{r}{a}) e^{i\omega t} \cos n\theta \tag{11a}$$

$$\psi = 0 \tag{11b}$$

$$\chi = C_{n2} J_n (\Omega \frac{r}{a}) e^{i\omega t} \sin n\theta \tag{11c}$$

It is immediately clear from Eqs. (2) that we obtain a plane strain mode of vibration wherein u_z vanishes identically. Substitution into Eqs. (2) and, thence, into the boundary conditions given by Eqs. (6) results in the following frequency equation:

$$J_{n-1} (\Omega) J_{n+1} (\kappa\Omega) + J_{n+1} (\Omega) J_{n-1} (\kappa\Omega) = 0 \tag{12}$$

wherein we have used the recurrence relations for the Bessel functions. These transcendental frequency equations define a doubly infinite set of natural circular frequency coefficients which depend upon Poisson's ratio. In Reference (22) this dependency is investigated in detail. The first four branches of the first four modes of vibration are plotted in function of κ which depends only upon Poisson's ratio. Additionally, the displacement fields for several different modes are plotted.

The last problem of interest concerning this initial crude model is that of transverse wave propagation. In Ref. (23) this problem was treated in some detail. The dispersion equations were derived and several branches were plotted. Additionally, it was pointed out that the dispersion equations degenerate for infinite wavelengths into two uncoupled frequency equations defining the axial-shear and transverse vibrations modes which we have discussed above.

Before proceeding to the refinement of our initial mathematical model, mention should be made of an interesting problem that arises in connection with Eq. (12), the frequency equation for free, transverse vibrations. It was mentioned that the natural frequency depends upon Poisson's ratio. In Ref. (22) it was demonstrated that finite, real natural frequencies exist when the material of the core is ideally incompressible; i.e., when Poisson's ratio has the value 1/2. A question immediately arises: "How can natural frequencies exist for an incompressible material when it occupies the entire internal volume of the tank?" In Ref. (24) it was demonstrated that, as the core material tends to become incompressible; i.e., as Poisson's ratio tends to 1/2, the frequency spectrum tends to a simple line spectrum. Clearly, this kind of a frequency spectrum is physically impossible. That this should be the case is not surprising since the ideally incompressible material is a hypothetical material which cannot exist in nature. Nevertheless, the assumption of such a material is quite regularly used in practice. The behavior mentioned above is simply one difficulty that can arise from the application of such an assumption. Finally, we draw attention to a few interesting studies due to Magrab (25, 26, 27) concerning the displacement and stress fields in the solid grain due to steady-state, forced harmonic oscillation of the rigid case.

On the basis of the studies cited we conclude that our initial crude model has been thoroughly investigated and that its steady-state response is adequately understood. Therefore, we proceed to a slight refinement of our mathematical model by allowing for a circular internal perforation of the grain. Now our model consists of an infinitely-long, two-layered cylinder with the outer layer ideally-rigid and the inner layer elastic. The boundary conditions in this case are given by the following:

$$\sigma_r \Big|_{r=b} = \tau_{r\theta} \Big|_{r=b} = \tau_{rz} \Big|_{r=b} = 0 \tag{13a}$$

$$u_r \Big|_{r=a} = u_\theta \Big|_{r=a} = u_z \Big|_{r=a} = 0 \tag{13b}$$

wherein a and b denote the external and internal radii, respectively, of the grain. Equations (13a) express the conditions that the internal grain perforation be free of surface tractions while Eqs. (13b) result from the assumption of an ideal bond between the grain and rigid casing.

For the axial-shear mode of vibrations we seek solutions in the form given by Eq. (7) so that Eq. (8) is the governing equation of motion while Eq. (9a) gives an admissible solution. In view of Eqs. (5), (7) and (9a) we find that all but two of the boundary conditions given by Eqs. (13) are trivially satisfied and these two conditions result in the following:

$$C_{n1} J_n (\Omega) + C_{n2} Y_n (\Omega) = 0$$
$$(n = 0, 1, 2\ldots)$$
$$C_{n1} J_n' (\Omega \frac{b}{a}) + C_{n2} Y_n' (\Omega \frac{b}{a}) = 0$$

This is a homogeneous system of linear, algebraic equations in the unknown constants C_{n1} and C_{n2}. Such a system can have non-trivial solutions only if the determinant of the coefficients of the unknowns vanishes identically. Thus, we are led to the following frequency equations:

Axisymmetric Modes:

$$J_o (\Omega) Y_1 (\Omega \frac{b}{a}) - J_1 (\Omega \frac{b}{a}) Y_o (\Omega) = 0 \qquad (14a)$$

Anti-symmetric Modes:

$$J_n (\Omega) Y_n' (\Omega \frac{b}{a}) - J_n' (\Omega \frac{b}{a}) Y_n (\Omega) = 0, \; n = 1, 2\ldots \qquad (14b)$$

The zeros of these frequency equations were calculated in Ref. (21).

In Ref. (28) the problems of plane strain vibrations and transverse wave propagation in the more refined model are treated. In the case of transverse wave propagation none of the boundary conditions given by Eqs. (13) is trivially satisfied so that a dispersion equation is obtained in the form of a 6 x 6 determinant set equal to zero. The elements of the determinant are, in general, linear combinations of n^{th}-order Bessel functions of the first and second kind. This dispersion equation is written in Ref. (28) and sample calculations were carried out in Ref. (29) for an incompressible grain. In Ref. (28) it is demonstrated that the dispersion equation governing transverse wave propagation degenerates for infinite wavelength to two uncoupled frequency

equations. One of these is that for axial-shear modes of vibration as given by Eq. (14b) and the other is that governing plane strain vibrations. The latter frequency equation is a 4 x 4 determinant involving n^{th}-order Bessel functions of the first and second kind set to zero. Four branches for each of the first four modes of vibration were calculated in Ref. (30) for an incompressible grain.

The next refinement we can make in our mathematical model of the rocket motor is to allow for an elastic casing so that our new model consists in a composite cylinder of two concentric elastic layers both of which are infinitely-long. In general, the stiffness of the grain will be considered small as compared with that of the casing material in deference to the origins of the problem. The understanding of the behavior of such a model should shed considerable light upon the rocket motor problem. To be sure, the outer layer which models the motor casing will usually be thin relative to its mean radius and, consequently, shell theory may be used in describing its behavior. However, for the time-being, we shall treat both layers by means of elasticity theory as given in Eqs. (1) - (5). We shall return, subsequently, to the use of shell theory in treating the behavior of the outer layer. In order to associate a given quantity with one or the other of the two layers, we append a 1 when referring to the outer layer and a 2 when reference to the inner layer is intended. Thus, u_{z2} denotes the axial component of displacement in the inner layer while u_{z1} denotes the same quantity in the outer layer. In accord with this convention, the boundary conditions are given by

$$\sigma_{r_1}\Big|_{r=r_1} = \tau_{r\theta_1}\Big|_{r=r_1} = \tau_{rz_1}\Big|_{r=r_1} = 0 \qquad (15a,b,c)$$

$$u_{r_1}\Big|_{r=a} = u_{r_2}\Big|_{r=a} \qquad (15d)$$

$$u_{\theta_1}\Big|_{r=a} = u_{\theta_2}\Big|_{r=a} \qquad (15e)$$

$$u_{z_1}\Big|_{r=a} = u_{z_2}\Big|_{r=a} \qquad (15f)$$

$$\sigma_{r_1}\Big|_{r=a} = \sigma_{r_2}\Big|_{r=a} \tag{15g}$$

$$\tau_{r\theta_1}\Big|_{r=a} = \tau_{r\theta_2}\Big|_{r=a} \tag{15h}$$

$$\tau_{rz_1}\Big|_{r=a} = \tau_{rz_2}\Big|_{r=a} \tag{15i}$$

$$\sigma_{r_2}\Big|_{r=r_2} = \tau_{r\theta_2}\Big|_{r=r_2} = \tau_{rz_2}\Big|_{r=r_2} = 0 \tag{15j,k,l}$$

For the axial-shear mode of vibration the only non-zero component of displacement is the axial component and we take it in the form

$$u_{z_j} = \left[C_{j1}\, J_n\left(\Omega_j \frac{r}{a}\right) + C_{j2}\, Y_n\left(\Omega_j \frac{r}{a}\right)\right] e^{i\omega t} \cos n\theta \tag{16a}$$

$$(j = 1,\ 2)$$

wherein

$$\Omega_j^2 = \rho_j \omega^2 a^2 / G_j, \quad (j = 1,\ 2) \tag{16b}$$

Direct substitution reveals that the equations of motion given by Eqs. (1) are identically satisfied for this displacement field. Substituting into Eqs. (5) we find that the only non-zero components of stress are the

following:

$$\tau_{rz_j} = \frac{G_j\Omega_j}{a} \left[C_{j1} J_n' (\Omega_j \frac{r}{a}) + C_{j2} Y_n'(\Omega_j \frac{r}{a})\right] e^{i\omega t} \cos n\theta \tag{17a}$$

$$(j = 1,2)$$

$$\tau_{\theta z_j} = -G_j \frac{n}{r} \left[C_{j1} J_n (\Omega_j \frac{r}{a}) + C_{j2} Y_n (\Omega_j \frac{r}{a})\right] e^{i\omega t} \sin n\theta \tag{17b}$$

We see immediately that 8 of the twelve boundary conditions given by Eqs. (15) are trivially satisfied and only the conditions given by Eqs. (15c,f,i,l) remain to be satisfied. Substitution into the latter conditions results in a homogeneous system of four linear, algebraic equations in the unknown constants C_{j1} and C_{j2}. The necessary and sufficient condition that non-trivial solutions of this system exist is that the determinant of the coefficients of the unknowns vanishes identically. Thus, we obtain the following frequency equations:

$$\begin{vmatrix} J_n'\left(\Omega_1 \frac{r_1}{a}\right) & Y_n'\left(\Omega_1 \frac{r_1}{a}\right) & 0 & 0 \\ J_n'(\Omega_1) & Y_n'(\Omega_1) & J_n'(\Omega_2) & Y_n'(\Omega_2) \\ R\alpha J_n(\Omega_1) & R\alpha Y_n(\Omega_1) & J_n(\Omega_2) & Y_n(\Omega_2) \\ 0 & 0 & J_n'(\Omega_2 \frac{r_2}{a}) & Y_n'(\Omega_2 \frac{r_2}{a}) \end{vmatrix} = 0 \tag{18a}$$

wherein $n = 0,1,2,\ldots.$ On expansion and rearrangement, these frequency equations reduce to the following:

$$f_{1n}\, h_{2n} - R\alpha h_{1n}\, f_{2n} = 0, \quad (n = 0,1,2,\ldots.) \tag{18b}$$

wherein

$$f_{jn} = J_n' (\Omega_j) Y_n' (\Omega_j \frac{r_j}{a}) - J_n' (\Omega_j \frac{r_j}{a}) Y_n' (\Omega_j) \qquad (19a)$$

$$J = 1, 2)$$

$$h_{jn} = J_n (\Omega_j) Y_n' (\Omega_j \frac{r_j}{a}) - J_n' (\Omega_j \frac{r_j}{a}) Y_n (\Omega_j) \qquad (19b)$$

$$R = \rho_2 / \rho_1 \qquad (19c)$$

$$\alpha^2 = \Omega_1^2 / \Omega_2^2 = \frac{G_2/G_1}{\rho_2/\rho_1} \qquad (19d)$$

The frequency equation given by Eq. (18) is an implicit function relating the density ratio ρ_2/ρ_1, the ratio of the shear modulii G_2/G_1, the radius ratios r_1/a and r_2/a and one or the other of the two frequency coefficients Ω_1 or Ω_2. In Ref. (31) it is shown that this frequency equation includes Eqs. (10) and (14) as special cases, as it should. Additionally, various branches of the frequency equation are plotted in order to establish the conditions under which the earlier simpler solutions can be used. In general, it would seem that when the casing is reasonably thin, the assumption of a rigid casing is acceptable only for axisymmetric, axial-shear modes.

For transverse (plane strain) vibrations of the composite cylinder we take the axial displacement to vanish identically and all displacements and stresses to be independent of the z coordinate variable. Under these conditions τ_{rz} and τ_z vanish identically and four of the twelve boundary conditions given by Eqs. (15) are trivially satisfied. Substitution into the remaining boundary conditions results in a system of 8 homogeneous, linear, algebraic equations in the 8 unknown constants. The necessary and sufficient condition that non-trivial solutions of this system exist is that the determinant of the coefficients of the unknowns must vanish. Thus, the frequency equation is obtained in the form of an 8 x 8 determinant set to zero. In general, the elements of the determinant are linear combinations of n^{th}-order Bessel functions of the first and second kinds. Calculations of natural frequencies have been carried out in Ref. (32) for a case wherein the outer layer (casing) is very much stiffer than the inner layer (grain). In this particular case, the various branches of the frequency equation have been plotted and analyzed for modes wherein n = 1; i.e., for modes that have only a single nodal diameter. Modes of vibration have been identified that degenerate to pure grain and pure casing modes. A pure grain mode is defined as that mode of vibration existing in the grain when the case is perfectly rigid while a pure

casing mode occurs in an empty casing. It was shown that, for typical geometries and material parameter values, the lowest $n = 1$ casing mode is substantially higher than the lowest $n = 1$ grain mode. It is demonstrated that the natural frequency of the lowest casing mode is very sensitive to grain thickness. It may be possible to include this effect in an approximate solution based on shell theory wherein the mass of the grain is lumped with the mass of the thin casing and the stiffness of the grain is neglected. This possibility should be exploited. Additionally, it was shown that the grain modes are relatively insensitive to variations in casing thickness indicating that, at least for the material parameters considered, the coupling of casing and grain rigidities is relatively weak. It follows that reasonable approximations to the natural frequencies of the grain modes can be obtained by assuming that the casing is perfectly rigid. The studies of Ref. (32) were limited to $n = 1$ modes. Work is in progress at The Catholic University of America to obtain similar results for $n = 2,3,4$ modes. The frequency equation has been programmed for machine computation, the program has been checked and data produced. It remains to analyze the data presently available and to supplement it as required.

As previously mentioned, it is possible to use thin shell theory in treating the deformation of the casing rather than elasticity theory. However, since thin shell theory is not substantially simpler than elasticity theory in this particular application, there appears to be little profit especially since elasticity theory must be used for the grain. Sann and Shaffer (33) have performed an interesting study wherein shell theory was used for the casing. Although the frequency equations are developed for all modes of transverse vibrations, numerical results and analysis are presented only for axisymmetric modes. Two different solutions are presented: one applicable to a compressible grain and the other to an incompressible grain. It is found that with a compressible core, the axisymmetric mode has two uncoupled motions; one is a 'rigid' rotation of the casing with a twisting of the grain and the other is a purely radial motion. The latter motion is not present in an incompressible grain. Some simplified frequency equations are also presented for limiting extremes of rigidity and density ratios.

The steady-state, transverse wave propagation problem for the two-layered, elastic cylinder remains for further treatment. None of the boundary conditions given by Eqs. (15) is trivially satisfied in this case so that the dispersion equation takes the form of a 12×12 determinant set equal to zero. This dispersion equation is readily written down but it remains to calculate its branches and perform the requisite analyses thereof.

We are once again ready for an additional refinement of our mathematical model. There are two types of refinements that can be made, both of which are fraught with difficulties: (1) complicated internal perforation and (2) finite length. It would appear that each of these possible refinements requires individual treatment.

Let us first consider the problem of the complicated geometry of the internal perforations of the grain. These internal passages are usually three-dimensional in character having shapes that are usually dictated by the considerations of internal ballistics. Little has been accomplished concerning the dynamics of such motors nor is it likely that a great deal will be achieved in the near future other than numerical solutions of specific problems. However, it frequently happens that these internal grain perforations are two-dimensional in nature over substantial portions of the total length of the motor. In these portions the grain cross-section is circular, of course, with a star-shaped perforation. Therefore, it is not unreasonable to consider an infinitely-long grain with such a cross-section. A considerable volume of work has been accomplished with this model of a rocket motor and in the following paragraphs we shall attempt to survey at least a portion of the work with which the author has a familiarity.

For a mathematical model consisting of an infinitely-long, circular grain with a complicated internal perforation, ideally bonded along its outer periphery to a rigid casing, the boundary conditions are that along the outer periphery the displacements must vanish while the periphery of the complicated internal perforation must be traction-free. The latter condition is expressed analytically as follows:

$$\begin{aligned} &\sigma_r n_r + \tau_{r\theta} n_\theta = 0 \\ &\tau_{r\theta} n_r + \sigma_\theta n_\theta = 0 \qquad \text{along } S(r,\theta) = 0 \\ &\tau_{rz} n_r + \tau_{\theta z} n_\theta = 0 \end{aligned} \tag{20}$$

wherein $S(r, \theta) = 0$ defines the internal perforation and n_r and n_θ denote the components in the radial and circumferential directions of the unit normal drawn outwardly with respect to the domain under consideration. In writing this condition we have taken n_z, the axial component of the unit normal, to vanish since the grain is cylindrical. Our initial considerations concerning this problem should be directed toward developing a method of solution. For this reason we choose to simplify the problem by considering a solid cylindrical bar with its outer periphery having the same shape as the internal perforation of the grain. Thus, the boundary condition applicable to this bar problem is given exactly by Eq. (20). Basically, the only difference between the bar problem and the grain problem is that the displacement boundary conditions are absent in the former. Consequently, techniques successfully applied in the bar problem should also be successful for the grain problem.

Additionally, for the sake of simplicity, we concern ourselves only with the axial-shear mode of free vibrations. Thus, we seek solutions in the form

$$\phi = \chi = 0 \tag{21a}$$

$$\psi = \Psi\,(r,\ \theta)\ e^{i\omega t} \tag{21b}$$

Accordingly, Eqs. (2) reduce to

$$u_r = u_\theta = 0 \tag{22a}$$

$$u_z = -e^{i\omega t}\ \nabla_1{}^2\ \Psi(r,\theta) \tag{22b}$$

and Eqs. (3) degenerate to the following single Helmholtz equation:

$$\nabla_1{}^2\ \Psi = -\frac{\rho\omega^2}{G}\ \Psi \tag{23}$$

Finally, as a consequence of Eqs. (22), the stress-displacement equations given by Eqs. (5) become

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = 0 \tag{24a}$$

$$\tau_{rz} = \rho\omega^2\ \frac{\partial \Psi}{\partial r}\ e^{i\omega t} \tag{24b}$$

$$\tau_{\theta z} = \rho\omega^2\ \frac{1}{r}\frac{\partial \Psi}{\partial \theta}\ e^{i\omega t} \tag{24c}$$

Now, it can be shown* that the outward unit normal to S in the plane of S is given by

$$\vec{n} = \vec{i}n_r + \vec{j}n_\theta = \frac{\vec{\nabla}S}{|\vec{\nabla}S|}$$

* See, for example, Wylie (34)

wherein

$$\vec{\nabla} S = \vec{i}\,\frac{\partial S}{\partial r} + \vec{j}\,\frac{1}{r}\frac{\partial S}{\partial \theta}$$

It follows that

$$n_r = \frac{\partial S}{\partial r} \Big/ \left| \vec{\nabla} S \right| \tag{25a}$$

$$n_\theta = \frac{1}{r}\frac{\partial S}{\partial \theta} \Big/ \left|\vec{\nabla} S\right| \tag{25b}$$

$$\left|\vec{\nabla} S\right| = \left[\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial S}{\partial \theta}\right)^2 \right]^{1/2} \tag{25c}$$

In view of Eqs. (24) and (25) the boundary conditions given by Eqs. (20) reduce to the following single condition:

$$\left[\frac{\partial \Psi}{\partial r}\frac{\partial S}{\partial r} + \left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta} \right)\left(\frac{1}{r}\frac{\partial S}{\partial \theta}\right) \right] \Bigg|_S = 0 \tag{26}$$

Thus, the problem has been reduced to one of finding a solution of Eq. (23) such that the boundary condition given by Eq. (26) will be identically satisfied.

Closed form solution of the problem posed above is not feasible so we shall be content with approximate solutions.

In Ref. (35) the collocation method is applied to solve the problem for a star-shaped boundary (with four star tips) given by

$$S(r,\theta) = r - a - b \cos 4\theta = 0 \tag{27}$$

We readily recognize the similarity of this family of plane curves to the boundary curve of the internal perforation of many common solid propellant rocket motors. Another advantage of this family is the fact that the circle is one curve of the family. Consequently, solutions can be degenerated to those for circular boundaries. Since the latter are available, we have a ready means for checking the results. The collocation method used consisted

in taking a solution in the form of a finite sequence of solutions of Eq. (23) and satisfying the boundary condition given by Eq. (26) at a finite number of points on the boundary. Little is known concerning convergence of the collocation method applied to eigenvalue problems such as the present case. It is generally presumed that, provided a sufficient number of collocation points are used, the resulting eigenvalues will be reasonably accurate. Even less is known concerning the manner of distributing the collocation points along the boundary although it is generally presumed that the distribution becomes less important as the number of collocation points increases. In this study the first four natural frequency coefficients were calculated and plotted in function of the parameter b/a which governs the length of the star tips. When b/a = 0, the bar is circular and as b/a increases the star tip grows longer. The study concludes that, for the problem under consideration, the collocation method is very sensitive to the distribution of collocation points. Furthermore, little convergence is demonstrated for as many as seven collocation points taken within an actant of the boundary. In view of the fact that it is not possible to obtain exact, closed form solutions in problems of this type, procedures for obtaining upper and lower bounds on the branches of the frequency equation are sorely needed. Only then can we be expected to make definitive statements concerning error in approximate procedures. Such bounding techniques frequently begin with estimates of the eigenvalues. Perhaps the value of the collocation method lies in its ability to provide these estimates fairly easily and quickly.

Jain (36) has introduced a new kind of collocation procedure which eliminates some of the difficulties of the boundary collocation method referred to above. He refers to the method as 'extremal point collocation'. He has applied the new method with striking success to the solution of several boundary value problems. The method requires the initial selection of a finite number of collocation points. This selection is arbitrary as in boundary collocation. Instead of requiring that the error in satisfaction of the boundary conditions vanish at the collocation points, as in boundary collocation, in extremal point collocation it is required that the error at adjacent collocation points be equal in magnitude but opposite in sign. Furthermore, the error at the collocation points must be larger than that at any other boundary point. It is from the latter condition that the method derives its name. In order to satisfay these collocation conditions, an iterative procedure is required by means of which the distribution of collocation points is determined. Extremal point collocation has two distinct advantages. Selection of the collocation points is not arbitrary; instead it is determined by the method. In ordinary collocation there is no control over the maximum error or the distribution of error. In extremal point collocation the error can nowhere exceed that at the collocation points and the error is uniformly distributed over the entire boundary.

In Ref. (37) the problem defined by Eqs. (23), (26) and (27) was solved using extremal point collocation. In this study it is shown that the method is an effective technique for the calculation of eigenvalues. It is relatively simple to use provided that a large-scale computer is available. The

method converges rapidly and yields reasonably accurate results even for a small number of collocation points. The question of accuracy requires further study by means of a method wherein the eigenvalues can be bounded.

Let us return to the boundary conditions given by Eqs. (20) for further consideration. If we substitute into Eqs. (20) from Eqs. (24), we obtain

$$\left(\frac{\partial \Psi}{\partial r} n_r + \frac{1}{r} \frac{\partial \Psi}{\partial \theta} n_\theta \right) \Bigg|_S = 0$$

The quantity on the left is the normal derivative which defines the rate of change of Ψ in the direction of the normal to S. Thus, the boundary condition that requires that the lateral surface of the bar be traction-free can also be written in the following form:

$$\frac{\partial \Psi}{\partial n} \Bigg|_S = 0 \tag{28}$$

This form suggests that there may be some advantage in formally mapping the bar cross-section onto a unit circle. If the conformal transformation is defined by

$$w = w\ (\xi)$$

wherein $w = re^{i\theta}$ defines the real plane while $\xi = Re^{i\mu}$ defines the complex plane, the boundary condition given by Eq. (28) becomes

$$\frac{\partial \Psi}{\partial R} \Bigg|_{R=1} = 0 \tag{29}$$

To be sure, satisfaction of this boundary condition is a trivial task compared to satisfaction of the boundary condition given by Eq. (28). However, we should hasten to point out that, by conformal transformation, we have simplified the task of satisfying the boundary conditions but we have complicated the task of finding solutions of the differential equation of motion since it, too, must be transformed. It can be shown* that the plane, Laplacian operator transforms according to

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} = \left| \frac{d\xi}{dw} \right|^{-2} \left(\frac{\partial^2 \Psi}{\partial R^2} + \frac{1}{R} \frac{\partial \Psi}{\partial R} + \frac{1}{R^2} \frac{\partial^2 \Psi}{\partial \mu^2} \right) \tag{30}$$

*See, for example, p. 629 of Wylie (34)

and it follows, therefore, that, under the conformal transformation, Eq. (23) becomes

$$\frac{\partial^2\Psi}{\partial R^2} + \frac{1}{R}\frac{\partial\Psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\mu^2} = -\frac{\rho\omega^2}{G}\left|\frac{d\xi}{dw}\right|^2 \Psi \qquad (31)$$

It is immediately clear that we will probably have difficulties finding solutions of this differential equation. Nevertheless, this formulation of the problem proves advantageous in certain circumstances. For example, common approximate methods of solution of problems of this type such as Rayleigh's principle or the Ritz method require trial functions that satisfy the boundary conditions. In the transform plane such trial functions can be formulated with little difficulty.

In Ref. (38) conformal transformation is used as outlined above in the solution of the problem of axial-shear vibrations of a star-shaped bar with a cross-section in the form of a four-lobed epitrochoid. The epitrochoidal boundary was chosen since its conformal mapping onto the unit circle is relatively simple and since it possesses the general characteristics of the solid propellant rocket grain perforation. To solve the problem in the complex plane the collocation method is applied wherein the boundary condition is identically satisfied and the error in satisfaction of the differential equation is collocated at a finite number of points in the interior of the unit circle. The results were considered favorable though some difficulty was encountered with regard to the spatial distribution of collocation points.

Among the many methods available for the solution of eigenvalue problems the methods of Rayleigh and Ritz are probably the most familiar. These methods yield upper bounds on the eigenvalues but, in the absence of exact values, it is difficult to estimate the error in the approximate solution. Temple (39) suggested a method for estimating the error in each stage of an iteration procedure directed toward the precise determination of the lowest eigenvalue. Temple's method can be interpreted as one which establishes the lower bound. A very powerful method for the determination of upper and lower bounds on all eigenvalues has been developed independently by Kohn (40) and Kato (41), as a generalization of Temple's method. This method was applied in Ref. (42) for the calculation of the natural frequencies in axial-shear vibrations of circular and epitrochoidal bars. The resulting bounds were extremely accurate for the fundamental natural frequency but the accuracy deteriorated somewhat for the high natural frequencies. However, it should be pointed out that the bounds can be improved at will if one is willing to expend the requisite additional effort.

In an eigenvalue problem it is required to find one or more constants λ, called eigenvalues, and corresponding functions β, called eigenfunctions, such that a differential equation

$$M\,[\beta] = \lambda N\,[\beta]$$

is satisfied throughout a domain D subject to certain boundary conditions on the boundary of D. In general, the domain D may be either a one- or a two-dimensional continuum. For many eigenvalue problems M [..] and N [..] are both linear, self-adjoint, positive-definite, defferential operators with the order of M greater than N. Under these conditions the eigenvalues are real and positive and the eigenfunctions form an orthogonal set. When the eigenvalue does not appear in the boundary conditions, the eigenvalue problem is called special provided that the operator N has the form

$$N [\beta] = g\beta$$

where g is a prescribed continuous function which is positive throughout the domain D. Thus, the governing differential equation for special eigenvalue problems becomes

$$M [\beta] = \lambda g\beta$$

We see immediately that Eqs. (23) and (31) have this form and, since the eigenvalue does not appear in the boundary condition, it is clear that the axial shear vibrations problem is a special eigenvalue problem in both the real and complex planes. Therefore, we can make use of the methods that have been developed for special eigenvalue problems. Collatz (43) has developed a procedure for obtaining upper and lower bounds in special eigenvalue problems. Using a trial function which satisfies the boundary conditions, but not necessarily the differential equation, a few simple operations are performed and the upper and lower bounds result. However, these bounds may not be very good unless the trial function is close to the exact solution of the problem. The fact that the method is relatively simple to apply for any given function indicates that it might become much more useful if a systematic procedure were added for "choosing" these trial functions. Such a procedure was developed by Appl and Zorowski (44). Another such method was developed in Ref. (45) and applied in the solution of the axial-shear vibrations problem for an epitrochoidal bar. The bounds obtained were adequate and subject to additional refinement but the effort involved is probably less than that required to obtain Kohn-Kato bounds.

We have discussed a number of techniques that should prove useful in the solution of the vibrations problem for the rocket motor model consisting of an infinitely-long, circular grain with a complicated internal perforation, ideally bonded along its outer periphery to a rigid casing. The more effective of these techniques require trial functions that satisfy the boundary conditions. It is clear, therefore, that a key requirement is a technique for conformally transforming the circular domain with a complicated perforation onto a circular annulus. Wilson (46) has developed such a technique

but it is limited to the case that the web fraction* is relatively small. For example, Arango (47) has shown that, when the perforation has four axes of symmetry, the error in mapping the external boundary increases rapidly when the web fraction increases beyond 0.5. When the web fraction reaches the value 0.9, the outer boundary has lost all semblance of a circle. This error arises due to the fact that Wilson treated the conformal transformation of an infinite domain with a hole into another such domain. Consequently, while the mapping function accurately transforms the internal boundary into the unit circle, the external boundary transforms only approximately into a circle. Rim and Stafford (48) have very recently presented a simple method of deriving approximate mapping functions in the form of low order polynomials which conformally transforms an annular region into one whose inner and outer boundaries are star-shaped and circular, respectively. The derivation is based on the Schwarz-Christoffel transformation. This method has the same accuracy problems in transforming the outer boundary as Wilson's method. This concern with the accuracy of the transformation of the external boundary is of substantial importance since, while the web fraction may be relatively small in the unburned solid propellant grain, it will increase toward unity as a consequence of the burning process. It should be clear, therefore, that it is essential to develop techniques for the conformal transformation of finite, doubly-connected domains.

Laura (49) has developed such a technique based on numerical integration of a pair of dual integral equations derived by Kantorovich and Muratov (50) for the purpose of conformal transformation of an arbitrary, finite, doubly-connected region onto a circular annulus. Laura demonstrates that, if the domain under consideration has one or more axes of symmetry and one of the boundaries is a circle, the system of two integral equations simplifies considerably. Several illustrative transformations are presented including the transformation of a domain which is typical of a solid propellant rocket motor. The accuracy of the technique is exceptional.

For purposes of illustration, let us consider a grain cross-section with 4 axes of symmetry. An octant of this cross-section is shown in Fig. 1 between the heavily-accented inner and outer boundaries. The outer boundary is a circle with a radius of 18 inches while the inner boundary has a maximum radius of 9-inches and a minimum radius of 3-inches. The inner and outer boundaries are further identified with the captions $R = 1.0$ and $R = b = 2.61$, respectively. A mapping function of the following form was used to conformally transform (approximately but with more than adequate accuracy) the domain between the inner and outer boundaries in Fig. 1 onto a circular

* Web fraction is defined as the ratio of the diameter of the circle circumscribing the inner boundary of a doubly connected region to the diameter of the circle circumscribing the outer boundary.

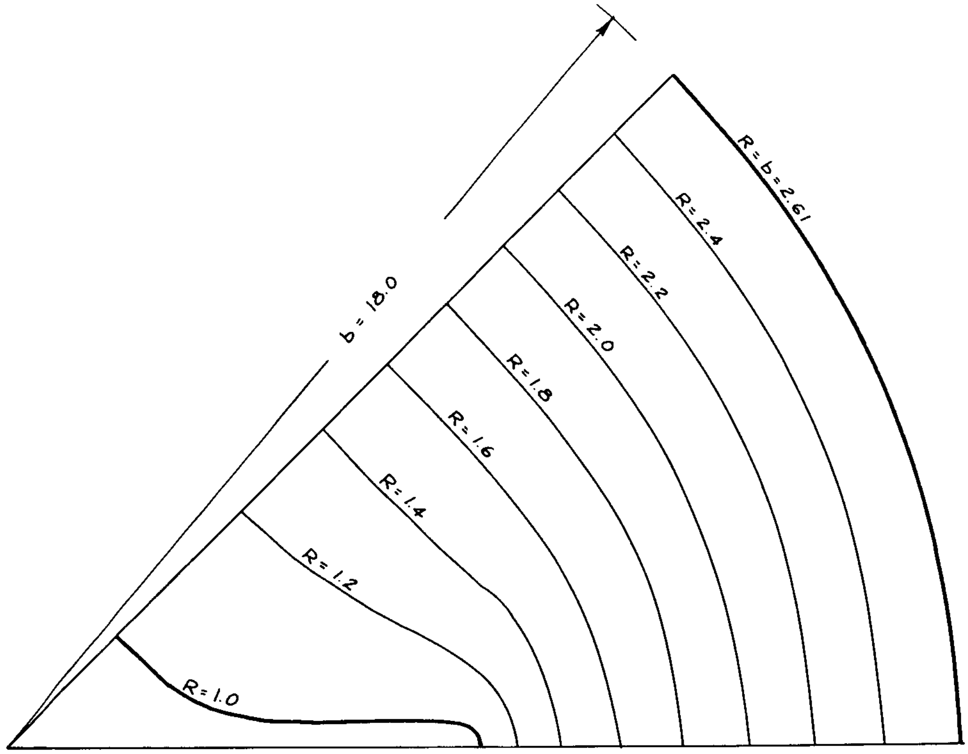


Fig. 1 First approximation to a set of burning curves for a typical solid propellant rocket motor

annulus with unit inner radius:

$$w = \sum_{j=0}^{N} A_{1-jp}\, \xi^{1-jp} + \sum_{j=1}^{M} A_{1+jp}\, \xi^{1+jp} \qquad (32)$$

wherein p denotes the number of axes of symmetry. In this case we had 4 axes of symmetry and, therefore, we took $p = 4$. A twelve term mapping function was used; i.e., we took $M = 1$ and $N = 10$. A larger number of terms would have resulted in a more accurate mapping. The unknown coefficients, $A_{1\pm jp}$ in Eq. (32) were evaluated by Laura's method. Using this mapping function seven additional circles in the complex plane, intermediate between the inner and outer boundaries, were transformed onto the curves shown in Fig. 1. Each of the contours shown is identified by the radius of the circle in the complex plane upon which it maps. Thus, the curve $R = 1.6$ transforms into the circle with radius 1.6, the curve $R = 2.0$ transforms into the circle with radius 20, etc. If one were able to stop the burning process in a solid propellant rocket motor at seven instants of time between ignition and burnout and to plot the shape of the inner boundary at each time, the so-called 'burning curves' would be obtained. The resulting contours would look much

like the curves of Fig. 1. Thus, as a first approximation, we regard the contours of Fig. 1 as a set of burning curves for a typical solid propellant rocket motor. Incidentally, it should be pointed out that, if actual burning curves were given, they could be conformally transformed onto circles in the complex plane with exceptional accuracy using Laura's method.

With the mapping function known it is not a difficult task to calculate the natural frequencies in the axial-shear mode for each of the regions of Fig. 1. The results would provide estimates of how the natural frequencies change during the burning process. There are any number of approximate procedures available for this calculation. We choose to use Galerkin's method. The applicable equation of motion is given by Eq. (31) which we choose to rewrite as follows:

$$\frac{\partial^2 \Psi}{\partial R^2} + \frac{1}{R}\frac{\partial \Psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2 \Psi}{\partial \mu^2} + \lambda^2 g\,(R,\mu)\,\Psi = 0 \tag{33a}$$

wherein

$$\lambda^2 = \frac{\rho\omega^2 b^2}{G} \tag{33b}$$

$$g\,(R,\mu) = \frac{1}{b^2}\left|\frac{d\xi}{dw}\right|^2 \tag{33c}$$

The quantity $g(R,\mu)$ was calculated using Eq. (32). The appropriate boundary conditions are

$$\Psi \Big|_{R=b} = 0 \tag{34a}$$

$$\frac{\partial \Psi}{\partial R}\Big|_{R=a} = 0 \tag{34b}$$

The first of these conditions expresses the fact that the displacement at the outer periphery of the grain must vanish since the grain is ideally bonded to a rigid case. The second condition above expresses the fact that the inner contour is free of tractions. The quantity a will take on 7 different values corresponding to the 7 different burning curves in Fig. 1. The following

trial function, satisfying the boundary conditions, was used:

$$\Psi(R,u) \approx \sum_{n=1}^{S} B_n W_n (R) = \sum_{n=1}^{S} B_n \left[\left(\frac{R}{B}\right)^2 - 2 \left(\frac{a}{b}\right) \left(\frac{R}{b}\right) + \left(2 \frac{a}{b} - 1 \right) \right]^n \tag{35}$$

We have chosen to ignore the μ -dependency, which we are at liberty to do since we are attempting an approximation. If the μ -dependency had been included, the resulting approximation would have been better than that obtained using Eq. (35). For these calculations we used a one-term trial function; i.e., S = 1. Substitution from Eq. (35) into Eq. (33a) results in

$$E\ (R,\mu) = \sum_{n=1}^{S} B_n \left[W_n'' (R) + \frac{1}{R} W_n' (R) + \lambda^2 g\ (R,\mu)\ W_n\ (R) \right] \tag{36}$$

wherein $\varepsilon(R,\mu)$ is an error function which does not, in general, vanish since, in general, W_n is not an eigenfunction. Galerkin's method requires that the error function $\varepsilon\ (R,\ \mu)$ be orthogonal to the trial functions $W_n(R)$ throughout the domain of interest; i.e.,

$$\int_D E(R,\ u)\ W_n\ (R)\ dD = 0, \quad (n = 1,2,\ldots\ldots,S) \tag{37}$$

Performing the indicated integration results in S homogeneous, linear, algegraic equations in the S unknown constants B_n. We obtain a frequency equation by requiring that the determinant of the coefficients of the unknowns vanish identically. This frequency equation was solved 7 times for the lowest natural circular frequency coefficient corresponding to the 7 different values of a. The results have been plotted in Fig. 2. The horizontal coordinate in Fig. 2 is the web fraction but, since the web fraction will be a function of time as the burning process proceeds, the plot in Fig. 2 shows the variation of the lowest natural circular frequency coefficient with time throughout the burning process from ignition to burnout. We see that the natural frequency increases rapidly as the motor burns out which is not an unexpected result. Such information is of extreme importance in the design of a guidance loop for the vehicle in which the motor is to be used.

It has been demonstrated that substantial progress has been made concerning the treatment of complicated grain perforations when these passages are substantially two-dimensional. Conformal transformation of the perforation by Laura's method should open the door to the use of a variety of acceptable methods for calculating natural frequencies in various modes of vibration and to the study of various steady-state wave propagation and forced vibrations problems. Incidentally, it is worth pointing out that Laura's method

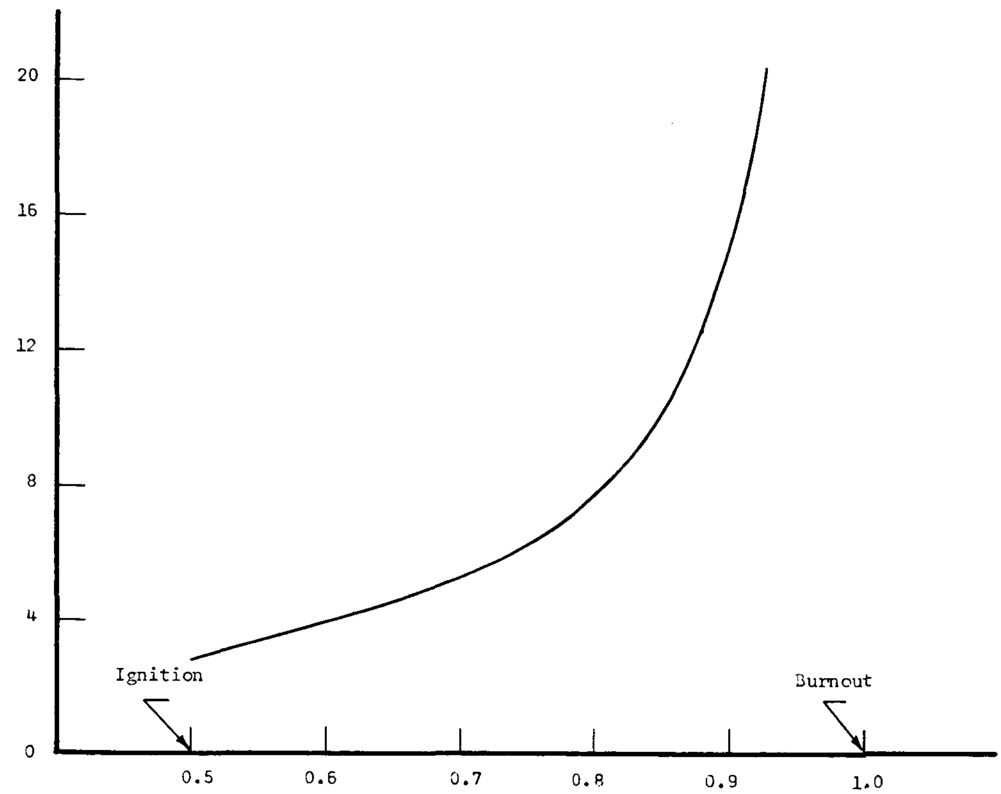


Fig. 2 Variation of the fundamental natural circular frequency coefficient in the axial-shear mode with web fraction for a typical solid propellant rocket motor

is equally applicable to the study of problems wherein inertial effects are unimportant such as quasi-static internal pressurization. Thus far only axial-shear modes of vibration have been considered for grains with complicated perforations. Additional effort is required on transverse modes and in wave propagation studies. When the geometry of the perforation varies rapidly with distance parallel to the motor axis, application of these two-dimensional methods becomes questionable. Little, if any, work is being accomplished on these three-dimensional problems. It would seem that future work in this area will be confined to the application of finite difference techniques.

We come now to the consideration of the ultimate refinement of our model of a solid propellant rocket motor; consideration of a model of finite length. Most of the modern solid propellant rocket motors are short with length-to-diameter ratios of the order of unity not uncommon. Little has been accomplished with the study of such dynamic problems. However, much is being done with quasi-static problems of this nature, particularly using finite difference schemes of various types. The development of understanding of finite

length effects and the study of the validity of our two-dimensional results when applied to relatively short motors are probably the most important unsolved problems in solid propellant motor dynamics today. Some little work is being accomplished but there is a noticeable lack of literature on the subject. We should call attention to the experimental study of the vibrations of short, solid elastic cylinders due to McMahon (51). Extensive experiments were performed and comparisons with available theory presented. However, the available theories were limited to thick disk and Timoshenko beam theories. It should be clear that much future effort should be expended in the area of finite length cylinders.

VISCOELASTIC GRAINS

The study of elastic grains has been reasonably active as can be judged from the detail of our discussion thereof. This is as it should be since, because of the elastic-viscoelastic correspondence principle,* solutions of elastic grain problems are intimately related to the solutions of viscoelastic grain problems. This principle states that the Laplace- or Fourier-transformed solution of a viscoelastic problem can be obtained from the Laplace- or Fourier-transformed solution of the associated elastic problem by replacing the elastic constants therein with appropriate functions of the Laplace or Fourier transform parameter. Therefore, in principle, it would seem that one can always obtain the viscoelastic solution associated with a given elastic solution. In practice, the situation is not quite so straight-forward since inversion of the viscoelastic solution remains to be accomplished. To be sure, exact inversion is always preferable when possible. In the practical situation this is rarely possible since the quantities replacing the elastic constants are usually measured material property functions of the Laplace or Fourier transform parameter available only in curve or tabular form. It should be clear, therefore, that a considerable effort should be mounted concerning approximate techniques for performing Laplace and Fourier transformations and inversions. In this regard we should mention the methods discussed by Schapery (56) and Cost (57) for the Laplace transform and by Solodovnikov (58) and Papoulis (59) for the Fourier transform. Assuming that approximate inversion methods will be available, our principal concern should be with elastic problems since these could be readily converted to viscoelastic solutions. Clearly, therefore, those difficulties that arise in elastic problems will remain with us in viscoelastic problems; for example, the finite length difficulty is still troublesome. Summarizing, it seems evident that viscoelastic problems present no fundamental difficulties that are not inherent in the associated elastic problems; the computational problem is more complicated and tedious but the basic qualities of the two types of problems remain the same.

*See Alfrey (52), Lee (53) and Sips (54, 55)

Because of the intimate relationship of elastic and viscoelastic problems and because of our rather lengthy discussion of elastic grain problems, we shall not belabor the viscoelastic problem area. However, we will mention a few pertinent investigations and present an illustrative solution, just enough to impart the flavor of the problem.

Gottenberg (60) has reported the results of an experimental investigation of steady-state, transverse vibrations of a long, steel cylindrical tube containing an inert propellant with a circular perforation. Many bending modes were detected and the resonant frequencies and mode shapes compared with the predictions of a Timoshenko beam theory in which the bending stiffness of the propellant was neglected relative to the casing stiffness but the additional mass of the inert propellant was included. The comparisons were quite adequate for engineering purposes. Substantial difficulties were encountered in detecting modes other than bending modes. However, one axisymmetric, longitudinal mode and a few breathing modes were identified. No theory was available for comparison with these modes.

Henry and Freudenthal (61) reported the results of an extensive analytical investigation of the steady-state, forced vibrations of a thick-walled viscoelastic cylinder contained by and bonded to a thin, cylindrical shell. Only axisymmetric solutions were considered. Complex frequency response functions were determined which may be easily used for arbitrary and random inputs by means of the well-established methods of generalized harmonic analysis. The study is broken down into three steps: (1) the thick-walled cylinder, (2) the thin shell, and (3) the composite cylinder. For a thick-walled, elastic cylinder the solutions of Eqs. (1) - (5) are developed in the usual manner expanding the normal tractions on the inner boundary and the normal and tangential tractions on the outer boundary in Fourier trigonometric series in the axial coordinate. The boundary conditions on the lateral surfaces are identically satisfied but, rather than requiring the axial normal and sheer stresses to vanish over the ends, the axial normal stress and the radial displacement are required to vanish. As a consequence of this relaxation of the boundary conditions, the physical significance of the results has been obscured to a certain extent. The difference between this 'finite-length' solution and the infinite-length solution is not clear. In order to obtain the viscoelastic solution, the elastic-viscoelastic correspondence principle is invoked wherein, for a forced vibrations problem, the elastic constants are replaced by complex material properties functions of frequency. Membrane theory is used for the thin elastic casing and a higher-order shell theory is included in an appendix. The latter should be used when the membrane theory is inadequate; for example, for external loads of acoustic nature. The assumption that the propellant and case form a continuous structure at their interface requires continuity of displacements and, therefore, produces strong coupling of the motions of grain and casing. This coupling is considered of primary importance and, therefore, the interaction is treated rigorously. The internal pressure is taken as harmonic and a large volume of numerical results is presented for the fundamental radial mode. It is concluded "that any analysis of a solid fuel rocket motor which

does not take into consideration the viscoelastic effects of the propellant would not give a true overall picture". Another important observation is arrived at by varying the grain geometry in simulation of the state of the composite structure at various stages of the burning process. The results indicate that, for certain ranges of frequency, there occurs a considerable increase in the amplitudes of stress and displacements. The implication is that the critical period in rocket operation would occur in the later stages of the burning process when the propellant is almost completely burned out. Thus, the important frequencies might be close to the fundamental radial mode of the casing.

In Ref. (62) the stress response to pressure transients has been investigated in an infinitely-long, two-layered cylinder having a thin, incompressible elastic outer layer and an inner layer of an incompressible, two-parameter Voigt material. This composite structure was taken as a crude mathematical model of a solid propellant rocket motor. The particular problem of interest concerned the circumferential stress response of the case to the pressure transients induced at ignition in the grain perforation. Two different pressure programs were studied in some detail: a square ignition pulse and a triangular ignition pulse followed by a pressure, constant in time. By solving the elastic problem first and then invoking the correspondence principle to obtain the viscoelastic solution, it was shown that the stress response is rather insensitive to the shape of the pulse. However, it was also demonstrated that the duration of the transient is important. When the duration of the transient is an order of magnitude smaller than the natural period of the cylinder in the radial mode, the stress response barely reflects the presence of the ignition pulse. However, as the duration of the transient approaches the natural period, the circumferential stress approaches the stress level corresponding to the pressure magnitude of the transient. The implication as far as casing and igniter design should be evident. If one is to economize on casing weight but nonetheless maintain rapid and positive ignition, the igniter must be designed such that the pressure transient will persist for a time no larger than one order of magnitude smaller than the natural period of the motor in the radial mode.

Lockett(63) presented the results of a study of significant importance with regard to the analysis of transient responses in viscoelastic materials. He discussed the effect produced by a rapid, but not discontinuous, change in pressure at the surface of a spherical cavity in an infinite, viscoelastic medium. Invoking the correspondence principle, Lockett obtains the solution to the viscoelastic problem from the Fourier transform of the associated elastic problem by replacing therein the elastic shear and bulk moduli with the viscoelastic complex shear and bulk modulus functions of circular frequency. Thus, he is subsequently concerned with performing inverse transformations. It is the manner of performing these inversions that is significant. Before proceeding to the actual inversion procedure, Lockett discusses the nature of modulus functions that should be used and the character of the pressure-time history. Most solutions to viscoelastic problems appearing in the literature are either

left in the integral form or are specialized to some simple spring-dashpot representation for the complex moduli. However, it has been shown by Kolsky and Shi (64) that, in general, these simple models do not adequately represent the behavior of real, viscoelastic materials. Thus, if only the solution to a particular problem is required, it would seem advisable to use the experimental data directly in the numerical inversion of the Fourier-transformed solutions. If it is desired to keep a number of parameters in the solution, then mathematical models should be chosen which fit the experimental data well, even though they may not correspond to a simple spring-dashpot configuration. The simple forms of modulus functions corresponding to spring-dashpot models only have an advantage in the solution of simple problems when it may be possible to evaluate the integrals explicitly. In his subsequent numerical work Lockett takes a constant bulk modulus; i.e., he assums that the material is elastic in dilatation, and for the complex shear modulus he uses an analytical expression that has been derived from curve fitting to experimental measurements performed on real, viscoelastic materials. For a pressure-time history Lockett selects a time function which has the following complex Fourier transform:

$$\bar{P}(\omega) = \begin{cases} 0, & 0 \leq \omega \leq \omega_1 \\ P_o / i\omega, & \omega_1 \leq \omega \leq \omega_2 \\ 0, & \omega_2 \leq \omega \leq \infty \end{cases} \tag{38}$$

wherein the range (ω_1, ω_2) defines the range in which accurate measurements are available of the complex shear modulus. It follows that, since the Fourier transform of the pressure-time history is a factor in the transformed solution, the exact solution in both the elastic and viscoelastic problems is a finite integral over the range (ω_1, ω_2) within which the shear modulus is accurately known. The following pressure-time history is obtained by inversion of the Fourier transform given by Eq. (38):

$$P(t) = \frac{P_o}{\pi} \int_{\omega_1}^{\omega_2} \frac{\sin \omega t}{\omega} \, d\omega = \frac{P_o}{\pi} \left[\mathrm{Si}\,(\omega_2 t) - \mathrm{Si}\,(\omega_1 t) \right] \tag{39}$$

where Si(x) denotes the sine integral which has been tabulated by Jahnke and Emde (65) and Abramowitz and Stegun (66), among others. For ω_2 very much larger than ω_1 this pressure program has the shape shown in Fig. 3. Lockett considers this loading in its own right and not as an approximation to an exactly square loading. The latter is not obtainable in practice, so the loading shown in Fig. 3 is more realistic. Obviously, the rise-time of the pressure program can be made shorter by increasing ω_2 up to the upper limit of the

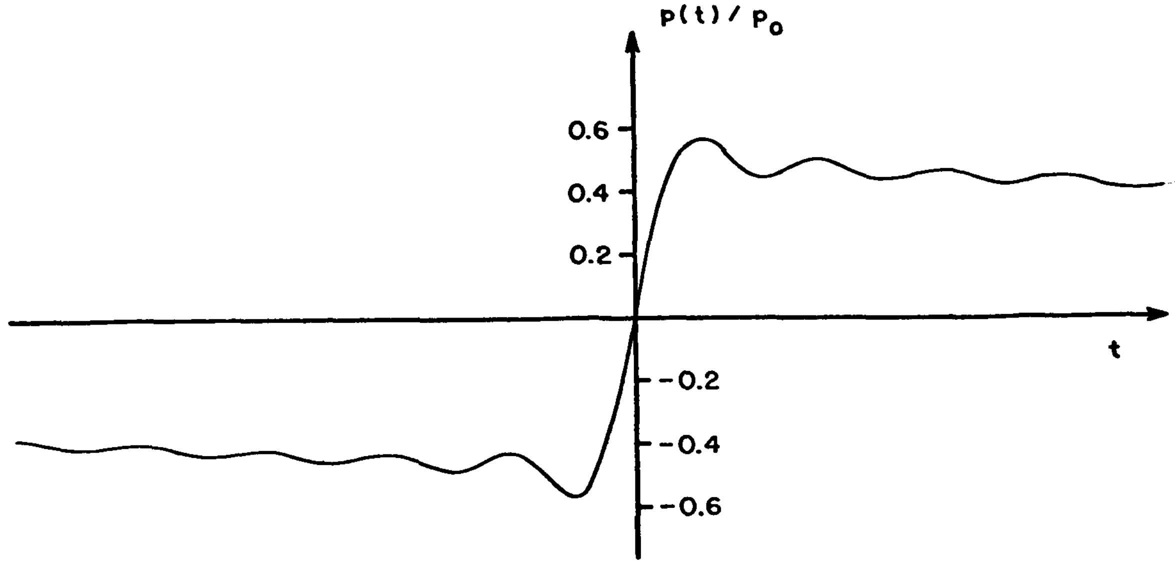


Fig. 3 First version of the alternate pressure program

frequency range in which the shear modulus is well-defined. The Fourier inversions of the transformed solutions are now performed numerically since integration is over a finite frequency range only. Lockett completes the solution and arrives at some interesting conclusions concerning the differences between elastic and viscoelastic solutions.

Making use of Lockett's suggestions the problem of dynamic internal pressurization was solved in Ref. (67) for an infinitely-long, incompressible, viscoelastic cylinder case-bonded to a thin elastic tank. The pressure-time history of Fig. 3 has some of the character of a pressure-time program of a solid propellant rocket motor but it has two important deficiencies: the applied pressure is negative for time less than zero and the pressure rise starts at negative time. Both of these deficiencies were corrected by appropriate shifting of axes so that Eq. (39) becomes

$$\frac{P(t)}{P_o} = \frac{1}{2} + \frac{1}{\pi} \left[\mathrm{Si}\ (\omega_2 t - \pi) - \mathrm{Si}\ \left(\omega_1 t - \pi\frac{\omega_1}{\omega_2} \right) \right] \tag{40}$$

This expression defines the one-parameter, ω_2/ω_1, family of curves shown in Fig. 4. We see that these curves possess most of the characteristics of actual pressure-time programs for solid rocket motors. We see a finite rise time which is controlled principally by the value of ω_2 and given very closely by

$$t_R = 2\pi/\omega_2 \tag{41}$$

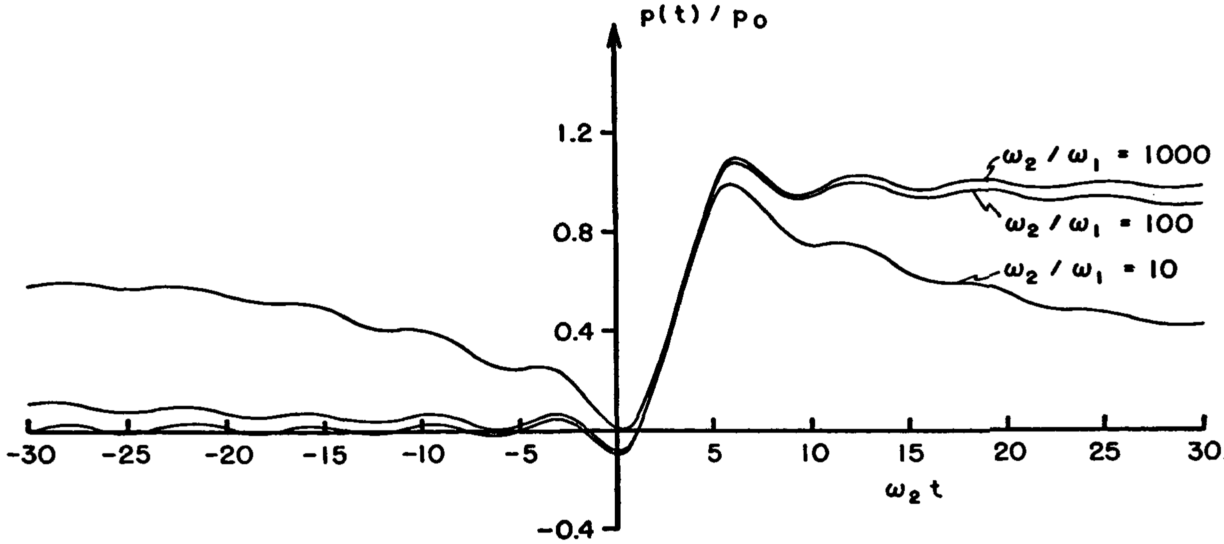


Fig. 4 Pressure-time program applied to the inner cylindrical surface of the core

The family of curves rises to a maximum and then oscillates about a quasi-static curve whose shape depends upon the value of the ratio $\omega_2/\ \omega_1$. In Ref. (67) a value of 1000 was used for this ratio and for this value the oscillation is about a curve that is approximately parallel to the dimensionless time axis at least for a substantial period of time. Since attention was focused on the immediate dynamic effects of the pressurization it mattered little that the pressure program decreases gradually for long times. The Fourier transform of the pressure program is given by

$$\frac{\bar{P}\ (\omega)}{P_o} = \pi\delta(\omega) + \frac{e^{-i\pi\omega/\ \omega_2}}{i\omega}\left[H\ (\omega-\ \omega_1) - H\ (\omega-\omega_2)\right] \tag{42}$$

This transform is non-zero over a limited frequency range only so that, ultimately, the inversion integrals can also be evaluated by numerical methods.

The shear storage and loss moduli used in the investigation were the actually measured data shown in Figs. 5. These material properties were introduced into the transformed solution by expressing the shear storage modulus in the following form:

$$G'(\omega) = G_o + \sum_{n=1}^{N} G_n \frac{\omega^2\tau^2 n}{1 + \omega^2\tau_n^2} \tag{43a}$$

wherein G_o, G_1, G_2,......, G_N, τ_1, τ_2,......., τ_N and N are arbitrary constants which must be evaluated such that Eq. (43a) will yield a good approximation to the shear storage modulus shown in Fig. 5(a) in the frequency range of interest. Equation (43a) is the analytical representation for the shear storage modulus for the (2N+1)-parameter Maxwell material. Thus, use of this expression implies that we have idealized the solid propellant as a linearly-viscoelastic, (2N+1)-parameter Maxwell material. It is clear in Eq. (43a) that G_o constitutes the shear storage modulus at zero frequency and, on the basis of extrapolation of the data of Fig. 3(a), the value selected was

$$G_0 = 400 \text{ psi}$$

The remaining parameters in Eq. (43a) were evaluated by means of a method introduced by Schapery (68) wherein the τ_n's were arbitrarily assigned and the G_n's evaluated by solving a system of N linear algebraic equations obtained by the collocation method. Five collocation points were used and the resulting analytical expression agreed with the curve of Fig. 5(a) to within the width of a pencil line. The analytical expression for the shear loss modulus for the (2N+1)-parameter Maxwell material, which is also required, is given by

$$G''(\omega) = \sum_{n=1}^{N} G_n \frac{\omega\tau_n}{1 + \omega^2\tau_n^2} \tag{43b}$$

It is clear, therefore, that the shear storage and loss moduli are not independent since the unknown parameters were evaluated from shear storage modulus data and these same parameters allow immediate calculation of the shear loss modulus by means of Eq. (43b). This result is not surprising and was recognized by Gross (69) when he calculated the following exact inter-relation between these two material property functions:

$$G''(\omega) = \frac{2}{\pi} \int_o^\infty G'(\alpha) \frac{\omega}{\alpha^2 - \omega^2} d\alpha \tag{44}$$

As an interesting experiment, the parameters calculated by means of Schapery's method using Eq. (43a) and the shear storage modulus data of Fig. 5(a) were used in Eq. (43b) to calculate the shear loss modulus. The results are plotted as a dotted curve in Fig. 5(b). We observe a very substantial disagreement between this result and the measured experimental data. This discrepancy has not been explained. Rather than using the experimental data of Fig. 5(b), Eq. (43b) was used for the shear loss modulus.

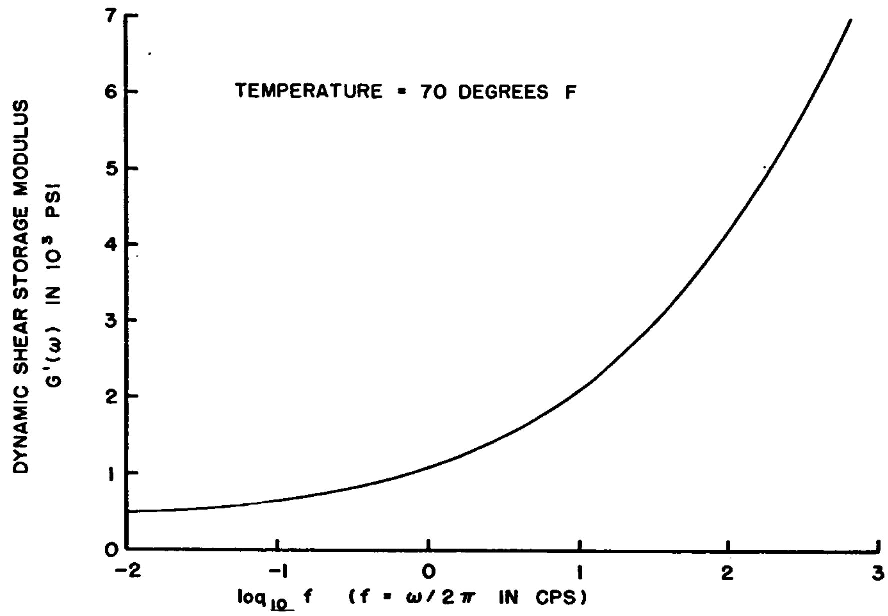


Fig. 5(a) Dynamic shear storage modulus as a function of frequency for Hercules Powder Co. CYH solid propellant

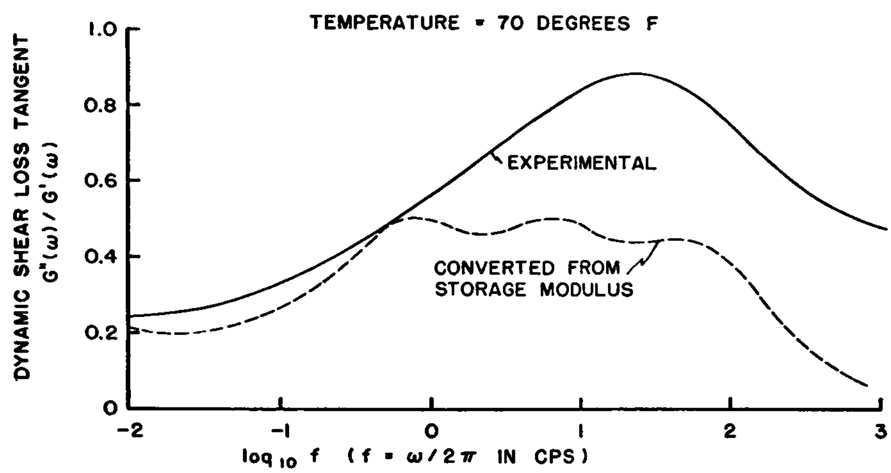


Fig. 5(b) Dynamic shear loss tangent as a function of frequency for Hercules Powder Co. CYH solid propellant

Making use of the transformed pressure program given by Eq. (42) and the material property functions shown in Eqs. (43) the transformed stresses and displacements in the two-layered cylinder were calculated. The inversion integrals were evaluated numerically with relative ease since they were integrals over a finite frequency range only. The results for the circumferential stress at the internal perforation are shown in Figs. 6. These results display some interesting features to which we should draw attention. First, we draw

attention to the small perturbations displayed by all solutions for time less than zero and for time greater than zero although, in the latter time regime, they may be obscured by larger oscillations. These perturbations of the calculated responses are clearly due to the small perturbations present in the pressurization program. These small oscillations can be ignored since our interest concerns the gross responses due to the gross pressure increase. Next, quasi-static solutions corresponding to all transient solutions were obtained (and plotted in Figs. 6) by taking the specific weights of both case and grain equal to zero. Now, we observe that, for the larger perforation (7.0-incher) the circumferential stress at the internal perforation is compressive despite the fact that the accompanying circumferential strains are tensile. See Figs. 6(a). This same effect was noted in Reference (70) and was explained therein. We summarize the explanation herein for purposes of completeness. The circumferential stress may be thought of as consisting of two components: a hydrostatic compressive component due to the compression of the core against the restraint offered by the case and a tensile component due to radial growth of the core under pressurization. If the core is sufficiently thin, such that the case restraint is important, then the compressive component is larger than the tensile component and the resulting circumferential stress is compressive even though the circumferential strain is tensile. This is undoubtedly the situation when a = 7.0 inches. It is clear that, with the appropriate combination of material properties and geometry, the opposite could also be true. As a matter of fact, we see from Fig. 6(b) that, for a = 1.3 inches and for both rise times, the circumferential stress at the internal perforation and the accompanying circumferential strain are both tensile. It seems obvious in this situation that case restraint plays the subordinate role.

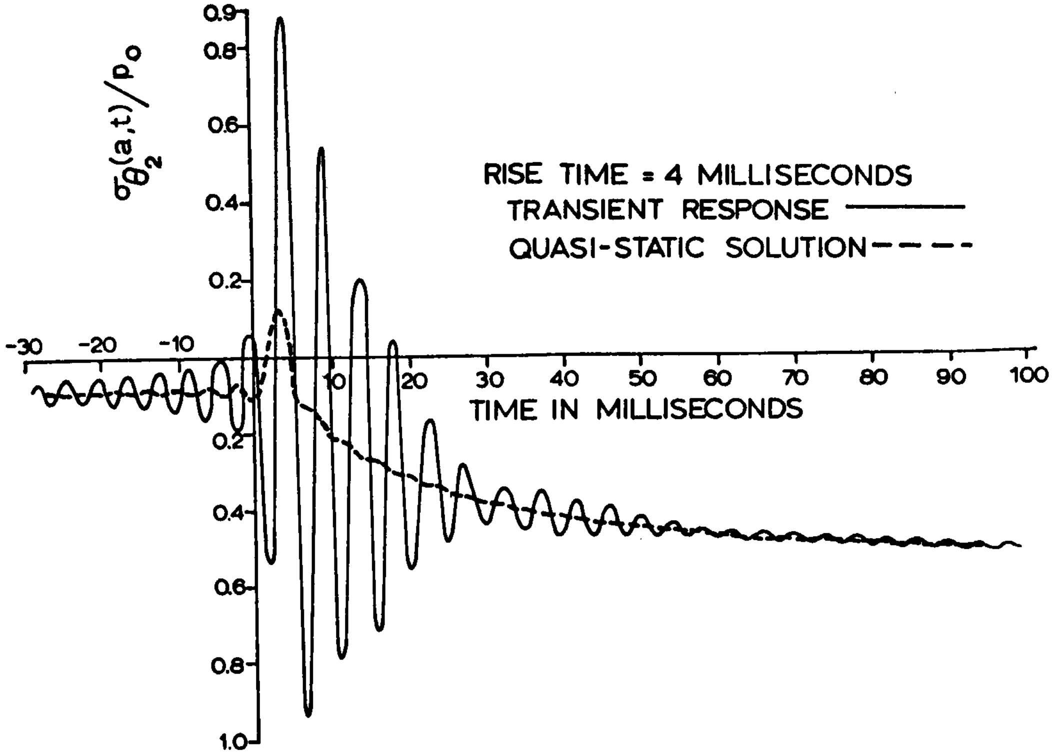
Now, we observe that for the 40 millisecond rise time the transient and quasi-static solutions coincided, for all practical purposes, for all times and in all cases considered. It seems clear that the 40 millisecond rise time is long compared to the fundamental natural period of the cylinder and, under these conditions, the transient and quasi-static formulations of the problem are equally valid thus accounting for the good agreement. However, for the shorter rise time we observe a substantial transient at small times and for the larger perforation. See Figs. 6(a). We also observe that the transient does not occur for the shorter rise time and for the smaller perforation. There appear to be two possible explanations of this behavior. For the larger perforation we have shown that the elastic casing plays a major role. The presence of this elastic storage element could account for initiation of the transient oscillation and for sustaining it thereafter. For the smaller perforation we have already demonstrated that the case plays a subordinate role and since there is a lack of the elastic storage element, the transient oscillation, if initiated at all, would attenuate very rapidly as a consequence of the dissipative quality of the core material which plays the major role. There is another candidate explanation. In Fig. 6(a) for the smaller rise time we see that the period of the transient oscillation, which is, of course, the fundamental natural period of the composite cylinder, has a value of about 4 or 5 milliseconds. Thus, since the rise time very nearly coincides

with the natural period of the cylinder, we expect to excite a substantial oscillation which will rapidly attenuate. For the smaller perforation, the natural period changes and the rise time no longer coincides and we do not expect to see a large transient oscillation. It is difficult, without extensive study, to choose between these explanations although we do favor the latter. There are, undoubtedly other features of these results that bear further disscussion but we feel that we have drawn attention to the salient features.

Achenbach (71, 72) has recently completed two investigations of general import. The first of these concerns the transient, dynamic response of an incompressible, elastic grain with an ablating circular port case-bonded to a thin, elastic case. Despite the fact that this study concerns an elastic grain, it is discussed here because of its close relationship with the second study. An internal pressure program in the form of a step function in time is applied and an approximate solution obtained by asymptotic integration of the equation of motion. It is demonstrated that burning time affects the period of the vibratory response but not the amplitude thereof. In the second investigation, Ref. (72), the dynamic response is calculated in an infinitely-long, composite cylinder consisting of a thich-walled cylinder of an incompressible, viscoelastic material case-bonded to a thin elastic shell. Two dynamic loadings are treated: a time step in internal pressure and a time step in external pressure. The latter loading condition is applied to an inoperative motor in a missile launching silo. The problem is solved, numerial results plotted and discussed for a grain of a standard, linear solid (three-parameter Maxwell material.) The manner of solution is discussed when measured data is used for the grain shear modulus.

We can judge by the character of these last few references that a great deal has been accomplished concerning transient, dynamic responses in viscoelastic grains. However, additional effort is required to perfect and simplify the application of the approximate methods for performing Fourier transforms and inversions. Then we can look forward to solving many transient dynamic problems using as input measured pressure programs and material property functions.

A great deal has been achieved in the solution of general wave propagation problems in linear viscoelasticity. See, for example, References (8), (73) - (91). This list of references is by no means complete nor is it a listing of the more important papers. Despite the fact that so much effort has been expended in investigating this general problem, thereby testifying to its importance, I was unable to locate a single reference involving a solution of direct applicability to solid propellant rocket motors. It seems evident therefore, that a considerable effort is required to elucidate those environments that give rise to wave propagation problems and in solving the problems posed.



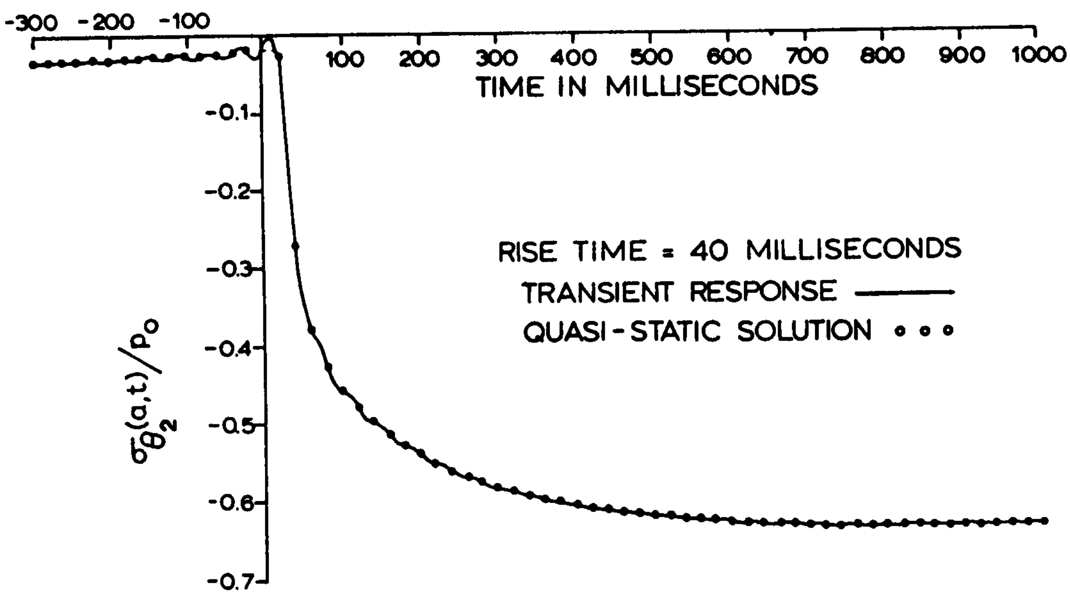


FIG. 6 (a) DIMENSIONLESS CIRCUMFERENTIAL STRESS RESPONSE AT INTERNAL PERFORATION (r = a) : a = 7.0 INCHES

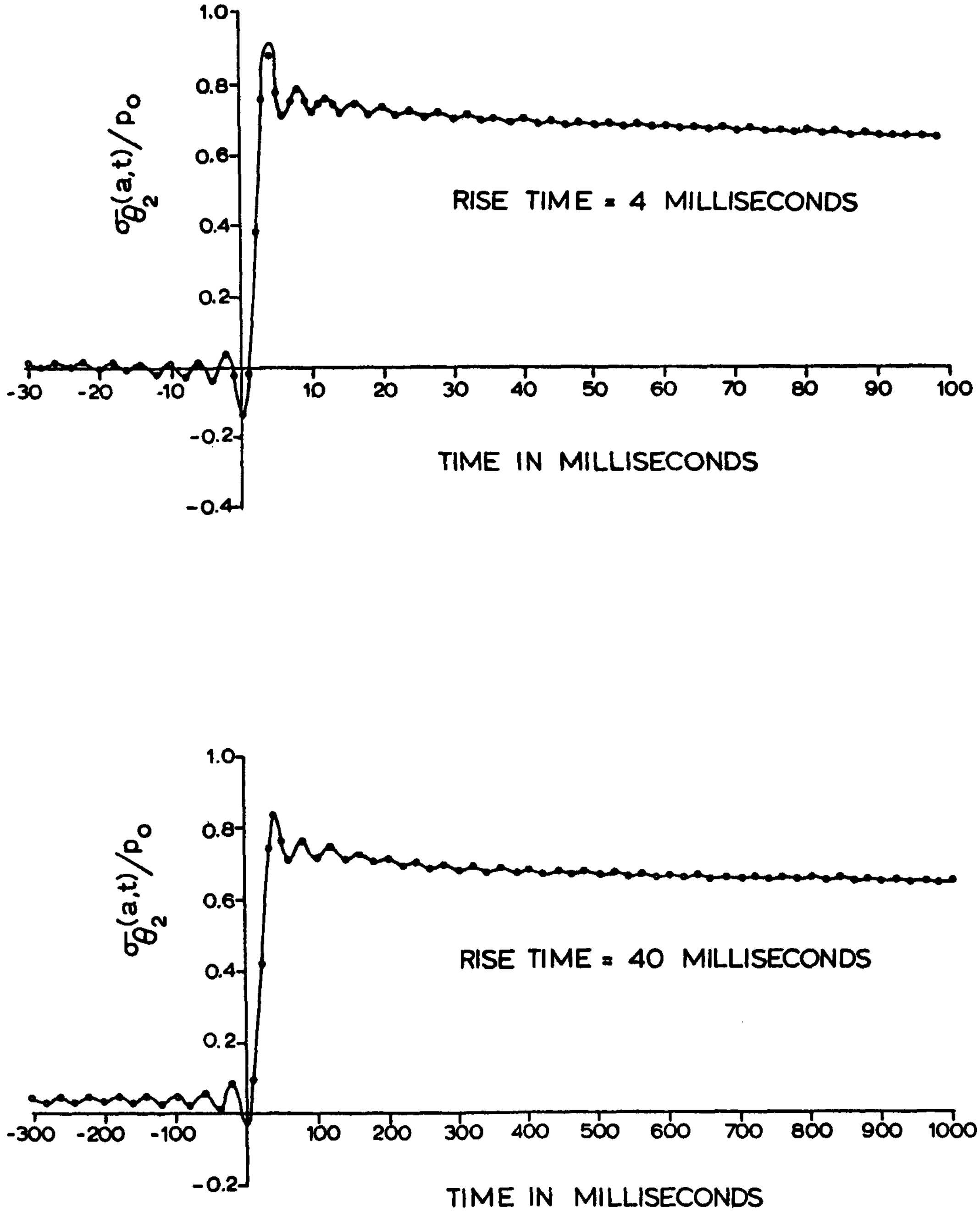


FIG. 6(b) DIMENSIONLESS CIRCUMFERENTIAL STRESS RESPONSE AT INTERNAL PERFORATION ($r = a$): $a = 1.3$ INCHES

TRANSIENT RESPONSE ———

QUASI - STATIC SOLUTION ∘∘∘

CONCLUSION

We have completed a survey of dynamic problems in solid propellant rocket motors. Attention was restricted to linear analysis of elastic and viscoelastic grains. It was not intended that this survey should be a bibliographical study but rather a presentation of progress in the field in an attempt to establish the current state-of-the-art. Many important related areas have been omitted in the interests of conservation of time and space. Consider, for example, nonlinear dynamic problems, shock problems, coupled thermo-mechanical problems, acoustic instability problems, and experimental measurement of propellant properties. Effort should be continued and expanded in these important areas among others. We have accomplished a great deal with regard to the dynamic problems of solid propellant rocket motors, however, additional effort is absolutely essential to extend the goals that we have already reached.

NOTATION

r, θ, z	radial, circumferential and axial coordinate variables of polar, cylindrical coordinate system
t	time
ρ	mass density
G	shear modulus
u_r, u_θ, u_z	radial, circumferential and axial components of displacement
Δ	cubical dilatation = $\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$
∇^2	Laplacian differential operator = $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$
ϕ, ψ, χ	displacement potential functions
c_c	dilatational wave velocity = $(G/\rho\kappa^2)^{1/2}$
c_s	shear wave velocity = $(G/\rho)^{1/2}$
κ	ratio of shear wave velocity to the dilatational wave velocity
$\sigma_r, \sigma_\theta, \sigma_z$	radial, circumferential and axial components of normal stress
$\tau_{r\theta}, \tau_{\theta z}, \tau_{rz}$	shear stress components
λ	Lame's second elastic constant = $2G\nu/(1-2\nu)$; dimensionless circular frequency coefficient
a, b, r_1, r_2	radii
w, Ω	circular frequency and circular frequency coefficient, resp.
n	mode number
J_n, Y_n	Bessel functions of the first and second kinds, resp., and of the n-th order

i $(-1)^{1/2}$

Primes over a variable denote ordinary derivatives of the variable with respect to its argument; e.g., $f'(r) = (d/dr)f(r)$

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